T The Vector Potential and Motion of Charged Particles in Axisymmetric Magnetic Fields

14p

Submitted for Publication

~~X64 11812~~

N65-89051

Code 2A

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(NASA TMX-51399)

Abstract

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Using the conventional expansion of a scalar magnetic potential (such as the earth's) an expansion of the vector potential is obtained. This expansion is used for analyzing the motion of charged particles in axisymmetric magnetic fields, with special attention to such fields that do not deviate far from a dipole. The results are compared to those of Quenby and Webber. Finally, the relation between Störmer's first integral and the third adiabatic invariant is traced.

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The Vector Potential

A curl-free magnetic field, such as that of the earth, is generally expressed by means of a scalar potential V

$$\underline{B} = - \operatorname{grad} V \tag{1}$$

Since V is harmonic, it is conveniently expanded in spherical harmonics

$$V(r,\vartheta,\varphi) = \frac{1}{R} \sum_{n,m} \left\{ a_{nm} \left(\frac{R}{r}\right)^{n+1} + b_{nm} \left(\frac{r}{R}\right)^{n} \right\} Y_n^m(\vartheta,\varphi)$$

$$Y_n^m(\vartheta,\varphi) = P_n^m(\vartheta) \ \exp\ im\varphi \tag{2}$$

where R is some constant length, e.g. the earth's radius.
Occasionally, however, it is useful to express $\underline{B}$ in terms of a vector potential $\underline{A}$

$$\underline{B} = \operatorname{curl} \underline{A} \tag{3}$$

If the scalar potential is given as in (2), $\underline{A}$ may be found in the following way. First of all, to reduce the arbitrariness in the choice of $\underline{A}$ the coulomb gauge condition is added

$$\operatorname{div} \underline{A} = 0 \tag{4}$$

$\underline{A}$ is then defined within the gradient of an arbitrary harmonic function and satisfies

$$\nabla^2 \underline{A} = 0 \tag{5}$$

Now it may be shown (Backus 1958) that any solenoidal vector $\underline{A}$ may be expressed by means of two scalars, Ψ_1 and Ψ_2, in the form

$$\underline{A} = \operatorname{curl} \Psi_1 \underline{r} + \operatorname{curl} \operatorname{curl} \Psi_2 \underline{r} \tag{6}$$

and the following identity holds

$$\operatorname{curl} \operatorname{curl} \Psi \underline{r} \equiv \operatorname{grad} \frac{\partial}{\partial r}(\Psi r) - \underline{r} \nabla^2 \Psi \tag{7}$$

In particular, if (5) is also satisfied, Ψ_1 and Ψ_2 may both be chosen to be harmonic (Smythe, 1950 ; § 7.04). Now if Ψ is a harmonic function,

$\partial(\psi r)/\partial r$ is one too and it is evident from (7) that Ψ_2 then adds to $\underline{A}$ only the gradient of a harmonic function and contributes nothing to $\underline{B}$. Using the remaining freedom in choice of $\underline{A}$, Ψ_2 may be set equal to zero, giving

$$\underline{A} = \text{curl}\ \Psi_1 \underline{r} \tag{8}$$

and by (7)

$$\underline{B} = \text{grad}\ \frac{\partial}{\partial r}(\Psi_1 r) \tag{9}$$

The last equation may be identified with (1). The vector potential is then given by (6), with

$$\Psi_1 = \frac{1}{R} \sum_{nm} \left\{ \frac{a_{nm}}{n}\left(\frac{R}{r}\right)^{n+1} - \frac{b_{nm}}{n+1}\left(\frac{r}{R}\right)^{n} \right\} Y_n^m(\vartheta,\varphi) \tag{10}$$

Axial Symmetry

From now on, only the case in which the field is axially symmetric, i.e. does not depend on φ, will be considered. For the time, however, $\underline{B}$ will not be restricted to be curl free. Then

$$B_r = \frac{1}{r^2 \sin\vartheta} \frac{\partial}{\partial\vartheta}(A_\varphi\, r \sin\vartheta) \tag{11a}$$

$$B_\vartheta = -\frac{1}{r\sin\vartheta}\frac{\partial}{\partial r}(A_\varphi\, r\sin\vartheta) \tag{11b}$$

If in addition

$$B_\varphi = 0 \tag{11c}$$

the index φ will be dropped from A_φ, for the vector potential

$$\underline{A} = \underline{i}_\varphi A \tag{12}$$

then, completely describes $\underline{B}$ as well as satisfying (4). The equation of a line of force in any meridional plane is then

$$\frac{dr}{B_r} = \frac{r\ d\vartheta}{B_\vartheta}$$

which with (11) gives

$$dr\, \frac{\partial}{\partial r}(A\, r \sin\vartheta) \quad + \quad d\vartheta\, \frac{\partial}{\partial \vartheta}(A\, r \sin\vartheta) = 0$$

or

$$A\, r \sin\vartheta = \alpha = \text{const}. \tag{13}$$

This equation has been obtained, from a somewhat different approach, by Ray (1963 ; bottom of p.9) . If the field is also curl-free, by (8)

$$A = -\frac{\partial \Psi_1}{\partial \vartheta}$$

Dropping the index m in (10) and using Legendre polynomials $P_n(\vartheta)$ gives

$$A = -\frac{1}{R} \sum_n \left\{ \frac{a_n}{n} \left(\frac{R}{r}\right)^{n+1} - \frac{b_n}{n+1} \left(\frac{r}{R}\right)^n \right\} \frac{dP_n}{d\vartheta} \tag{14}$$

which upon substitution in (13) gives the relation between r and ϑ on a line of force.

<u>Motion of a Charged Particle</u>

Consider a particle of rest mass m_0, charge q and velocity $\underline{v}$ moving in an axisymmetric field. Its Lagrangian will be (MKS)

$$L = -m_0 c^2 (1 - v^2/c^2)^{\frac{1}{2}} \quad + \quad q\, (\underline{v} \cdot \underline{A}) \tag{15}$$

Due to symmetry , φ is a cyclic coordinate. Denoting

$$m = m_0 (1 - v^2/c^2)^{-\frac{1}{2}}$$

the following first integral is obtained

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m\, r^2 \sin^2\vartheta\, \dot\varphi + q\, A_\varphi\, r \sin\vartheta = \text{const}. \tag{16}$$

Since energy is conserved, it is useful to divide (16) by $P = m\, v$ and to denote the new constant by γ . If ω is the angle between $\underline{i}_\varphi$ and $\underline{v}$, then

$$\cos\omega = \dot\varphi\, r \sin\vartheta / v$$

so that (16) becomes

$$\cos\omega = \frac{\gamma}{r\sin\vartheta} - \frac{q\,A_\varphi}{P} \tag{17}$$

This equation is a generalization of Störmer's (1955) treatment of the dipole field. It was used by Treiman (1953) in calculating effects of a ring current around the earth and also by Lüst and Schlüter (1957) who derived it directly from the equation of motion.

$$\frac{d}{dt}(m\,\underline{v}) = q\,(\underline{v} \times \underline{B}) \tag{18}$$

Let now (11c) be assumed, so that A_φ becomes A. Then (17) gives

$$(P/q)\,r\sin\vartheta\,\cos\omega = (\gamma P/q) - A\,r\sin\vartheta$$

If the particle's energy is low enough for the guiding-center approximation to hold, $\cos\omega$ will oscillate rapidly around zero and the particle's orbit in the (r, ϑ) plane will alternate between the two sides of the line of force

$$A\,r\sin\vartheta = \gamma P/q \tag{19}$$

This may be regarded as the particle's guiding line of force (for a similar approach, see Ray [1963]). One notes that in (17), $|\cos\omega|$ is always less than unity while - in a near-dipole field - $(q\,A_\varphi / P)$ may be made as large as one wants by going to low enough momenta. Thus at low momenta, the left hand side of (17) must be the difference between two much larger terms and the particle does not stray far from the line-of-force of (19).

In addition to (17) , Treiman (1953) also derived a method of calculating cut-off momenta (in the cosmic-ray sense,i.e. a criterion for finding when orbits are completely trapped by the field.) applicable to fields which do not deviate to far from a dipole field.When this is used,the following results are obtained.Assuming no external field sources ($b_n = 0$) , denoting the dipole moment by M and defining the Störmer unit of length

$$R_o = (q M \mu_o / P)^{\frac{1}{2}}$$

it is found that for given P (and consequent R_o) only trapped orbits exist when

$$r < R_1 \cong R_o - \frac{q}{2P} \sum_{n=3}^{\infty} a_n \left(\frac{R}{R_o}\right)^n \frac{dP_n}{d\vartheta}(\pi/2) \tag{20a}$$

$$\gamma > \gamma_c \cong 2R_o - \frac{q}{P} \sum_{n=3}^{\infty} \frac{a_n}{n} \left(\frac{R}{R_o}\right)^n \frac{dP_n}{d\vartheta}(\pi/2) \tag{20b}$$

The vertical cut-off momentum for orbits reaching the sphere $r = R$ at colatitude ϑ is then

$$P_c \cong q M \mu_o \left[\frac{\sin^2\vartheta}{2R} + \frac{\sin}{2M\mu_o} \sum_{n=2}^{\infty} \frac{a_n}{n} \left\{ \frac{\sin^{2n-1}\vartheta}{2^n} \frac{dP_n}{d\vartheta}(\pi/2) - \frac{dP_n}{d\vartheta}(\vartheta) \right\} \right]^2 \tag{21}$$

Comparison with the Quenby - Webber theory

One may compare these results to those obtained by Quenby and Webber (1959) , who used Treiman's method to obtain geomagnetic cut-off momenta but in addition introduced various approximations.The validity of these,especially as applied to non-axisymmetric fields, will not be discussed and the comparison will be restricted, of necessity, to symmetrical fields. For the vicinity of the equatorial plane Quenby and Webber assumed a vector potential in the φ direction and approximated its magnitude A. Modifying slightly the expression given for it by Webber (1963 , eq. 9) and changing the notation to that used here

$$A = \frac{M\mu_0 \sin\vartheta}{r^2} + \sum_{n=2} \frac{R^{n+2}}{r^{n+1}} \frac{\Delta B_n}{n} \sin\vartheta \tag{22}$$

Here $\Delta B_n(\vartheta)$ is the value, at $r = R$, of the horizonthal component of that part of $\underline{B}$ which falls off as $r^{-(n+2)}$. The first term in the expression gives the vector potential due to the main dipole and will not be further considered. The value of ΔB_n may be derived from the n-th component of the scalar potential V which is assumed to have no external sources ($b_n = 0$)

$$\Delta B_n = - a_n R^{-2} \frac{dP_n}{d\vartheta}$$

Equation (22) thus becomes

$$A = \frac{M\mu_0 \sin\vartheta}{r^2} - \frac{1}{R} \sum \frac{a_n}{n} \left(\frac{R}{r}\right)^{n+1} \frac{dP_n}{d\vartheta} \sin\vartheta$$

This differs from (14) only by the factor $\sin\vartheta$ which, in the vicinity of the equatorial plane, is close to unity. Equation (20b) may also be rewritten

$$\gamma_c \cong R_0 \left\{ 2 + \sum_{n=3} \frac{1}{n} \frac{\Delta B_n(\pi/2)}{\Delta B_1(\pi/2)} \left(\frac{R}{R_0}\right)^{n-1} \right\} \tag{23}$$

Webber (1963) obtains a similar equation (loc.cit.,eq.11) but without the factor $(R/R_o)^{n-1}$. These discrepancies suggest that the theory of Quenby and Webber may need modification. The expression for cut-off momenta given by that theory differs in its general form from (21) and will not be compared.

Adiabatic Invariance

Lagrange's equations and therefore eq.(16) still hold when the axisymmetric magnetic field is time dependent, even though energy is no longer conserved on account of the induced electric field. The same argument leading to the neglect of cos ω in (17) for low momenta then shows that, for low momenta, the second term of (16) is much larger than the first. Neglecting the first term completely gives

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} \cong q\, A_\varphi\, r \sin\vartheta \cong \text{const.} \tag{24}$$

Equation (24) shows, in a time dependent axisymmetric field, how low energy particles shift from one magnetic shell to another : the line-of-force parameter α of eq.(13) is then conserved. This result may be generalized as follows.

Suppose the axisymmetric field undergoes a perturbation which is now not only time dependent but also asymmetrical. Equation (16) and its low energy limit (24) then no longer hold. However, since the motion of trapped particles in the unperturbed field may be regarded as periodic in the coordinate φ (at least for reentrant orbits), the action integral

$$J_\varphi = \int_0^{2\pi} p_\varphi \, d\varphi$$

is adiabatically conserved (compare Landau & Lifshitz [1951] ,p.54),where p_φ is a component of the canonical momentum and may be approximated for low momenta by (24). The element of arc length is

$$\underline{dl} = \underline{i}_r\, dr + \underline{i}_\vartheta\, r\, d\vartheta + \underline{i}_\varphi\, r \sin\vartheta\, d\varphi$$

so that

$$J_\varphi \cong q \oint \underline{A}\, \underline{dl}$$

with the integral extending over one rotation of φ . By Stokes' theorem

$$J_\varphi \cong q \int \operatorname{curl} \underline{A}\, \underline{ds} = q\,\Phi$$

where Φ denotes the flux enclosed by the shell to which the particle is attached. Thus one obtains the flux invariant , or third adiabatic invariant (Northrop and Teller,1960) as a generalisation of Störmer's integral : in a perturbed axisymmetric field,the magnetic flux through a magnetic shell is adiabatically conserved.

Appendix

Treiman's approach was the following. By (17), for any given γ and v the accessible region in the (r, ϑ) plane is bounded by lines where $\cos\omega$ equals 1 or (-1). Of these (in fields not deviating much from a dipole) the former

$$\frac{\gamma}{r \sin\vartheta} - \frac{q A}{P} = 1 \tag{25}$$

determine whether trapping occurs. Regarding γ as a parameter, Treiman (1953) showed that trapping just starts when, in the equatorial plane, eq. (25) acquires a double root for r. In near-dipole fields without external sources, this occurs when, for $\sin\vartheta = 1$,

$$\frac{\partial \gamma}{\partial r} = 0$$

(when external sources exist this may not hold [Ray, 1956]). For purposes of calculation it is useful here to split A into two parts, A_1 giving the dipole field and A_2 (small by comparison) the higher terms. If M is the dipole moment

$$A_1 = \frac{M \mu_0 \sin\vartheta}{r^2} \tag{26}$$

Putting $\vartheta = \pi/2$ and neglecting all external sources, eq. (25) becomes

$$\gamma = r + \frac{q M \mu_0}{P} \frac{1}{r} - \frac{q}{P} \sum_{n=2} \frac{a_n}{n} \left(\frac{R}{r}\right)^n \frac{dP_n}{d\vartheta}(\pi/2) \tag{27}$$

Since $dP_n/d\vartheta$ vanishes in the equatorial plane for any even n, only odd values of n need to be considered in the last term. Let (25) be satisfied at $r = R_1$; then

$$\frac{q M \mu_0}{P} \frac{1}{R_1^2} - \frac{q}{P R} \sum_{n=3} a_n \left(\frac{R}{R_1}\right)^{n+1} \frac{dP_n}{d\vartheta}(\pi/2) = 1 \tag{28}$$

As a first approximation, let the higher terms be neglected. Then

$$R_1 \cong R_o = (q M \mu_o / P)^{\frac{1}{2}} \tag{29}$$

R_o is the well-known Störmer unit of length.Let now

$$R_1 = R_o (1 + \delta) \tag{30}$$

collecting all first-order terms in (25) gives

$$\delta \cong - \frac{q}{2 P R} \sum_{n=3} a_n \left(\frac{R}{R_o}\right)^{n+1} \frac{dP_n}{d\vartheta}(\pi/2) \tag{31}$$

However,substituting (30) in (27) shows that to the first order of approximation the critical γ (denoted γ_c) does not depend on δ .

$$\gamma_c = 2R_o - \frac{q}{P} \sum_{n=3} \frac{a_n}{n} \left(\frac{R}{R_o}\right)^{n} \frac{dP_n}{d\vartheta}(\pi/2) \tag{32}$$

It does,however,depend on the momentum P (assume for simplicity all particles are identical,e.g.protons) both directly and through R_o , and represents the limit of complete trapping for this momentum.Suppose now that such marginally trapped particles hit the earth ($r = R$) vertically ($\cos\omega = 0$) at colatitude ϑ ; they then represent the vertical cut-off momentum P_c at that colatitude and by (17)

$$\gamma_c = \frac{q\, r \sin\vartheta}{P} \left[\frac{M \mu_o \sin\vartheta}{R^2} - \frac{1}{R} \sum_{n\, 2} \frac{a_n}{n} \frac{dP_n}{d\vartheta} \right] \tag{33}$$

Neglecting nondipole components,one obtains from (32) and (33) as a first approximation

$$R / R_o = \tfrac{1}{2} \sin^2\vartheta \tag{34}$$

This approximation is inserted into the correction terms of (32),(33),giving

$$P_c \cong M q \mu_o \left[\frac{\sin^2\vartheta}{2R} + \frac{\sin\vartheta}{2 M \mu_o} \sum_{n=2} \frac{a_n}{n} \left\{ \frac{\sin^{2n-1}\vartheta}{R^n} \frac{dP_n}{d\vartheta}(\pi/2) - \frac{dP_n}{d\vartheta}(\vartheta) \right\} \right]^2 \tag{35}$$

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